

## THE CUTOFF PROFILE FOR THE SIMPLE EXCLUSION PROCESS ON THE CIRCLE

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In this paper, we give a very accurate description of the way the simple exclusion process relaxes to equilibrium. Let  $P_t$  denote the semi-group associated the exclusion on the circle with  $2N$  sites and  $N$  particles. For any initial condition  $\chi$ , and for any  $t \geq \frac{4N^2}{9\pi^2} \log N$ , we show that the probability density  $P_t(\chi, \cdot)$  is given by an exponential tilt of the equilibrium measure by the main eigenfunction of the particle system. As  $\frac{4N^2}{9\pi^2} \log N$  is smaller than the mixing time which is  $\frac{N^2}{2\pi^2} \log N$ , this allows to give a sharp description of the cutoff profile: if  $d_N(t)$  denote the total-variation distance starting from the worse initial condition we have

$$\lim_{N \rightarrow \infty} d_N \left( \frac{N^2}{2\pi^2} \log N + \frac{N^2}{\pi^2} s \right) = \operatorname{erf} \left( \frac{\sqrt{2}}{\pi} e^{-s} \right),$$

where  $\operatorname{erf}$  is the Gauss error function.

**1. Introduction.** The exclusion process is a lattice interacting particle system where particles perform independent nearest-neighbor random walks with the added constraint that each site can be occupied by at most one particle (see the classic references [13] and [14] for a complete introduction to the subject). It is a very simplified model for a gas of interacting particles. We consider in this [Introduction](#) the case were the lattice is either a  $d$ -dimensional torus or hypercube of side length  $N$ . The number of particle is chosen to be proportional to the number of sites.

In this paper, we investigate how the particle system starting far away from equilibrium, relaxes to its equilibrium state. This question can in fact be treated with different point of views:

- One can describe the evolution of the system at the macroscopic level, giving the evolution of the density of particle in space and time. This

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is the study of hydrodynamic limits (see [7] for an introduction to the subject).

- One can adopt a microscopic point of view, and look at the evolution of the law of the particle, and in particular, its total variation distance to the equilibrium law. This is the study of the Markov chain's mixing time (see [12]).

With the first point of view, it is now well known that under diffusive rescaling (space rescaled by  $N$  and time rescaled by  $N^2$ ), the density of particle evolves like the solution of the heat equation. The result is valid in any dimension (see [7] for references).

Concerning the mixing-time approach, progresses are more recent. It has been shown by Morris that in any dimension the time needed to come close to equilibrium in total variation was of order  $N^2 \log N$  [16]. In dimension 1, more refined estimates have been obtained and gave the exact location of the mixing time either for the segment [8] or the circle [9] with lower bounds proved earlier by Wilson [19] (see also [17] for results in the case of arbitrary graph, and [6, 10]). A natural question is then what does the law of the particle system look like when it is about to reach equilibrium.

At equilibrium, the law of the distribution is uniform over all particle configurations. Another way to see it is to say that the state of each site (occupied or vacant) is given by a field of i.i.d. Bernoulli variables whose sum is conditioned to be equal to the number of particle (which is a fixed parameter).

What would be natural to expect then, is that up to a small correction, before equilibrium, the particle distribution still is a conditioned product measure, but that the Bernoulli variables are no more identically distributed: there is a space dependent bias which is given by the solution of the heat equation. This brings a strong connection between the problem of the mixing time and that of the hydrodynamical limit. This connection was previously underlined by Lee and Yau when studying the related issue of log-Sobolev constant for the simple exclusion [11]. Indeed in the case of small bias, with some minor efforts one can derive sharp estimates on the total-variation distance between the conditioned product of biased Bernoulli and the equilibrium measure. This can be turned into a precise prediction on how the total-variation distance drops from one to zero, the cutoff profile. The present paper brings this heuristic picture on a rigorous ground in the case of the exclusion on the circle.

## 2. Model and results.

2.1. *The process.* We consider  $\mathbb{Z}_{2N} := \mathbb{Z}/(2N\mathbb{Z})$ , the discrete circle with  $2N$  sites and we place  $N$  particles on it, with *at most* one particle per

site. With a slight abuse of notation, we will sometimes use elements of  $\{1, \dots, 2N\} \subset \mathbb{Z}$  to refer to elements of  $\mathbb{Z}_{2N}$ .

The exclusion process on  $\mathbb{Z}_{2N}$  is a dynamical evolution of the particle system which can be described informally as follows: each particle tries to jump independently on its neighbors with transition rates  $p(x, x+1) = p(x, x-1) = 1$ , but the jumps are canceled if a particle tries to jump on a site which is already occupied.

Let us describe the chain more formally. We adopt the convention that 1 denotes a particle and  $-1$  denotes an empty site. This is not the most usual one (empty sites are more often denoted by 0) but it proves to be more practical in our computations. Our state-space is defined by

$$(2.1) \quad \Omega_N = \left\{ \eta \in \{-1, 1\}^{\mathbb{Z}_{2N}} \mid \sum_{x=1}^{2N} \eta(x) = 0 \right\}.$$

Given  $\eta \in \Omega$  define  $\eta^x$  the configuration obtained by exchanging the content of site  $x$  and  $x+1$

$$(2.2) \quad \begin{cases} \eta^x(x) := \eta(x+1), \\ \eta^x(x+1) := \eta(x), \\ \eta^x(y) = \eta(y) \quad \forall y \notin \{x, x+1\}. \end{cases}$$

The exclusion process on  $\mathbb{Z}_{2N}$  with  $N$  particle is the continuous time Markov process on  $\Omega_N$  whose generator is given by

$$(2.3) \quad (\mathcal{L}_N f)(\eta) := \sum_{x \in \mathbb{Z}_{2N}} f(\eta^x) - f(\eta).$$

The chain is irreducible and reversible, and the unique invariant probability measure is the uniform probability measure on  $\Omega_N$  which we denote by  $\mu_N$ . Given  $\chi \in \Omega_N$  we let  $(\eta_t^\chi)_{t \geq 0}$  denote the trajectory of the Markov chain starting from  $\chi$ . We write  $\mathbb{P}[(\eta_t^\chi)_{t \geq 0} \in \cdot]$  for the law of  $(\eta_t^\chi)_{t \geq 0}$ . We let  $P_t$  denote the Markov semi-group and write  $P_t^\chi$  for the probability measure  $P_t(\chi, \cdot)$ ,  $\chi \in \Omega_N$ .

We measure the distance to equilibrium in terms of total variation distance. If  $\alpha$  and  $\beta$  are two probability measures on  $\Omega$ , the total variation distance between  $\alpha$  and  $\beta$  is defined to be

$$(2.4) \quad \|\alpha - \beta\|_{\text{TV}} := \frac{1}{2} \sum_{\omega \in \Omega} |\alpha(\omega) - \beta(\omega)| = \sum_{\omega \in \Omega} (\alpha(\omega) - \beta(\omega))_+,$$

where  $x_+ = \max(x, 0)$  is the positive part of  $x$ . It measures how well one can couple two variables with law  $\alpha$  and  $\beta$ . We define the distance to equilibrium of the Markov chain to be

$$(2.5) \quad d^N(t) := \max_{\chi \in \Omega_N} \|P_t^\chi - \mu\|_{\text{TV}}.$$

In a previous paper [9], we described in detail the asymptotic behavior of  $d^N(t)$ . We proved that around a time of order  $\frac{N^2}{2\pi^2} \log N$  the distance to equilibrium drops from 1 to 0 in a time window of width  $N^2$ ,

$$(2.6) \quad \begin{aligned} \lim_{s \rightarrow \infty} \limsup_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log N + N^2 s \right) &= 0, \\ \lim_{s \rightarrow -\infty} \liminf_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log N - N^2 s \right) &= 1. \end{aligned}$$

The aim of this paper is to complete this picture by identifying, in an acute way, the pattern of relaxation to equilibrium. In particular, we are interested in proving the existence and finding an expression for limiting profile

$$(2.7) \quad \lim_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log N + N^2 s \right).$$

To reach this goal, we have to understand what the distribution  $P_t^\chi$  looks like much before the time  $\frac{N^2}{2\pi^2} \log N$ .

**2.2. The mixing time profile.** The main achievement of our paper is to determine the cutoff profile.

**THEOREM 2.1.** *The total-variation distance to equilibrium from the worst initial condition has the following asymptotic profile: for any  $s \in \mathbb{R}$ , we have*

$$(2.8) \quad \lim_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log N + \frac{N^2}{\pi^2} s \right) = \operatorname{erf} \left( \frac{\sqrt{2}}{\pi} e^{-s} \right),$$

where  $\operatorname{erf}$  is the Gauss error function

$$(2.9) \quad \operatorname{erf}(t) := \frac{2}{\sqrt{\pi}} \int_0^t e^{-u^2} du.$$

The method by which we obtain the result gives us in fact much more information about the relaxation of the system: we are able to characterize fully how  $P_t^\chi$  looks like much before the mixing time, for all initial condition  $\chi \in \Omega_N$ .

**REMARK 2.2.** The fact that the profile of the cutoff is given by a function of the type  $\operatorname{erf}(Ae^{-s})$  (the constant is not essential since it depends on the particular process and the choice for the normalization) is given by Wilson [19] as a conjecture (supported by numerical evidences) for a process very much related to the exclusion: the adjacent transposition shuffle. The reason why the function  $\operatorname{erf}$  appears is that the last statistic that comes

to equilibrium for the process (here the first Fourier coefficient of  $\eta$ , see below) is well approximated by a Gaussian; the exponential terms are present because the mean of this Gaussian converges exponentially to zero. This is a property which is believed to be shared by many Markov chains and rigorously known, for example, the random walk on the hypercube [5]. Let us mention that however there are the Markov chains which exhibit cutoff and do not have this property. This is, for instance, the case of top to random shuffle [4], and also of the transposition shuffle for which the important statistic, the number of fixed point, behaves like a Poisson variable (see, e.g., [15]).

*2.3. The description of  $P_t^\chi$  much before equilibrium.* The main result of the paper, from which we deduce Theorem 2.1 requires some notation to be introduced. The time evolution of the density of particles is given by the discrete heat equation on  $\mathbb{Z}_{2N}$  and for this reason, the eigenfunction of the discrete Laplacian on the circle plays an important role in our analysis; in particular, those in the eigenspace corresponding to the spectral gap:  $x \mapsto \cos(\frac{\pi x}{N})$ , and  $x \mapsto \sin(\frac{\pi x}{N})$ .

To describe the projection of  $\chi \in \Omega_N$  on this eigenspace, it is more convenient for us to have one positive coefficient than two real ones, and for this reason we introduce  $\theta(\chi)$  which is the “phase” of  $\chi$  in the first eigenspace. It is the unique  $\theta \in [0, 2\pi)$  satisfying

$$(2.10) \quad \begin{aligned} \sum_{x \in \mathbb{Z}_{2N}} \chi(x) \cos\left(\frac{\pi x}{N} + \theta\right) &= 0, \\ \sum_{x \in \mathbb{Z}_{2N}} \chi(x) \sin\left(\frac{\pi x}{N} + \theta\right) &> 0, \end{aligned}$$

or  $\theta(\chi) = 0$  if the system has no solution. We denote by  $b(\chi)$  the first Fourier coefficient of  $\chi$

$$(2.11) \quad b(\chi) := \frac{1}{N} \sum_{z \in \mathbb{Z}_{2N}} \chi(z) \sin\left(\frac{\pi z}{N} + \theta\right).$$

Note that  $b(\chi) = 0$  in the case where (2.10) has no solution. In the case where  $\chi = +1$  for  $x \in \{1, \dots, N\}$  and  $-1$  elsewhere,  $\eta(\chi) = \frac{\pi}{2N}$ .

If  $\mu$  is a probability measure on a state-space  $\Omega$  and that  $f$  is a function  $\Omega \rightarrow \mathbb{R}$ , we use the following notation for the expectation:

$$(2.12) \quad \nu(f) := \nu(f(\eta)) := \sum_{\eta \in \Omega} f(\eta) \nu(\eta).$$

We define given  $N$ ,  $\alpha > 0$  and  $\theta \in [0, 2\pi)$ . We define  $\nu^{N,\alpha,\theta}$  to be the probability measure on  $\Omega_N$  with the following Radon–Nikodym density:

$$(2.13) \quad \frac{d\nu^{N,\alpha,\theta}}{d\mu_N}(\eta) := \frac{e^{\alpha a_\theta(\eta)}}{\mu_N(e^{\alpha a_\theta(\eta)})},$$

where

$$(2.14) \quad a_\theta(\eta) := \sum_{x \in \mathbb{Z}} \eta(x) \sin\left(\frac{\pi x}{N} + \theta\right).$$

Finally, let us set

$$(2.15) \quad \lambda_N := 2\left(1 - \cos\left(\frac{\pi}{N}\right)\right).$$

We note that  $\lambda_N$  is the spectral gap of the simple random walk on  $\mathbb{Z}_{2N}$  (with jump rate one in each direction), and hence from [3], Section 4.1.1, it is also the spectral gap of the exclusion process on  $\mathbb{Z}_{2N}$ .

The main result of the paper tells us that much before the mixing time,  $P_t^\chi$  is close to  $\nu^{N,\alpha,\theta}$  for an appropriate choice of  $\alpha$  and  $\theta$ .

**PROPOSITION 2.3.** *For all  $N$  sufficiently large, for all  $\chi \in \Omega_N$  for all  $t \geq \frac{4\pi^2}{9N^2}$ , we have*

$$(2.16) \quad \|P_t^\chi - \nu^{N,b(\chi)e^{-\lambda_N t},\theta(\chi)}\|_{\text{TV}} \leq (\log \log N)^{-1}.$$

Theorem 2.1 follows from Proposition 2.3 by using the following lemma.

**LEMMA 2.4.** *For all  $K > 0$ , for all  $N$  sufficiently large (depending on  $K$ ),*

$$(2.17) \quad \lim_{N \rightarrow \infty} \sup_{\substack{\gamma \in [0, K] \\ \theta \in [0, 2\pi)}} \left| \|\nu^{N,\gamma N^{-1/2},\theta} - \mu_N\|_{\text{TV}} - \text{erf}\left(\frac{\gamma}{\sqrt{8}}\right) \right| = 0.$$

**2.4. Exclusion with an arbitrary number of particle.** We have chosen to present here the result only in the case where the number of particles is equal to half of the number of sites. However, *mutatis mutandis*, the proof adapts to the case of  $k_N$  particle  $k_N \leq N$  on  $\mathbb{Z}_{2N}$  where  $k_N$  tends to infinity with  $N$  (the case  $k \geq N$  can be treated by symmetry). Let us discuss here what the results are in that case and how they can be obtained.

While the case of  $k_N$  proportional to  $N$  can be derived directly from the proof presented here, it turns out that some of the technical lemmas (e.g., Proposition 6.3) breaks down if  $k_N$  grows much slower, that is, like

$\log N$ . However, adapting the techniques developed specifically for the case of slowly growing  $k_N$  in [9], all technical obstacles can be overcome.

To close this discussion, let us mention what the cutoff profiles are in those cases. When  $k_N = \lceil \alpha N \rceil$  for some  $\alpha = (0, 1)$ , we have

$$(2.18) \quad \lim_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log N + \frac{N^2}{\pi^2} s \right) = \operatorname{erf} \left( \frac{\sin(\alpha\pi/2)}{\pi \sqrt{\alpha(1-(\alpha/2))}} e^{-s} \right).$$

When  $k_N$  satisfies  $1 \ll k_N \ll N$ , we have

$$(2.19) \quad \lim_{N \rightarrow \infty} d^N \left( \frac{N^2}{2\pi^2} \log k_N + \frac{N^2}{\pi^2} s \right) = \operatorname{erf} \left( \frac{1}{2} e^{-s} \right).$$

**2.5. Organization of the paper.** In Section 3, we prove Theorem 2.1 from Proposition 2.3, and also give a proof of Lemma 2.4. In Section 4, we decompose the proof of Proposition 2.3 into three key statements, whose proofs are, respectively, given in Sections 5, 6 and 7.

### 3. The cutoff profile.

**3.1. Proof of Theorem 2.1.** Let  $s \in \mathbb{R}$  be fixed. It is straightforward to check that for

$$t_{s,N} := \frac{N^2}{2\pi^2} \log N + \frac{N^2}{\pi^2} s$$

we have

$$(3.1) \quad \sup_N \sup_{\chi \in \Omega_N} \sqrt{N} b(\chi) e^{-\lambda_N t_{s,N}} < \infty.$$

Hence, using the triangular inequalities, Proposition 2.3 and Lemma 2.4 we have for all  $\chi \in \Omega_N$ ,

$$(3.2) \quad \lim \left| \|P_{t_{s,N}}^\chi - \mu_N\|_{\text{TV}} - \operatorname{erf} \left( \frac{b(\chi) \sqrt{N} e^{-\lambda_N t_{s,N}}}{\sqrt{8}} \right) \right| = 0.$$

The asymptotic for  $d_N(t_{s,N})$  follows if one can identify  $\chi$  which maximizes  $b(\chi)$ . A few seconds of thoughts are enough to realize that  $\chi_{\max}$  defined as follows is the unique maximizer up to translation:

$$(3.3) \quad \chi_{\max}(x) = \begin{cases} +1, & \text{for } x = 1, \dots, N, \\ -1, & \text{for } x = N+1, \dots, 2N. \end{cases}$$

The asymptotic behavior of  $b(\chi_{\max})$  is given by the following limit:

$$(3.4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{x \in \mathbb{Z}_{2N}} \chi_{\max}(x) \sin \left( \frac{x\pi}{N} - \frac{\pi}{2N} \right) = \frac{4}{\pi}.$$

As for any  $s \in \mathbb{R}$ , we have also

$$(3.5) \quad \lim_{N \rightarrow \infty} \sqrt{N} e^{-\lambda_N t_{s,N}} = e^{-s}$$

the result follows from the continuity of the error function.

3.2. *Proof of Lemma 2.4.* The underlying idea is quite simple: we want to prove that asymptotically under  $\mu_N$ , once rescaled

$$a_\theta(\eta) := \sum_{x \in \mathbb{Z}} \eta(x) \sin\left(\frac{\pi x}{N} + \theta\right),$$

converges to a Gaussian.

LEMMA 3.1. *The following statements hold true:*

(i) *For a fixed  $\theta \in [0, 2\pi)$ . The quantity  $N^{-1/2}a_\theta(\eta)$  converges in law to a standard Gaussian. Moreover, the convergence is uniform in  $\theta$ , in the sense that for any bounded continuous function  $F$  the convergence*

$$(3.6) \quad \lim_{N \rightarrow \infty} \mu_N \left[ F\left(\frac{a_\theta(\eta)}{\sqrt{N}}\right) \right] = \frac{1}{\sqrt{2\pi}} \int F(u) e^{-u^2/2} du,$$

*holds uniformly in  $\theta$ .*

(ii) *Moreover,  $a_\theta(\eta)$  is exponentially concentrated in the sense that there exists a constant  $c > 0$  such that*

$$(3.7) \quad \mu_N(|a_\theta(\eta)| \geq \sqrt{N}u) \leq 2e^{-cu^2}.$$

Let us explain how we deduce Lemma 2.4 from these facts. We note that

$$(3.8) \quad \|\nu^{N, \gamma N^{-1/2}, \theta} - \mu_N\|_{\text{TV}} = \frac{1}{2} \mu_N \left( \left| \frac{e^{\gamma N^{-1/2} a_\theta(\eta)}}{\mu_N(e^{\gamma N^{-1/2} a_\theta(\eta)})} - 1 \right| \right).$$

Because of convergence in probability and exponential tightness, we have

$$(3.9) \quad \lim_{N \rightarrow \infty} \mu_N(e^{\gamma N^{-1/2} a_\theta(\eta)}) = e^{\gamma^2/2}.$$

Thus,  $\|\nu^{N, \gamma N^{-1/2}, \theta} - \mu_N\|_{\text{TV}}$  converges uniformly in  $\gamma \in [0, K]$  and in  $\theta$ , to

$$(3.10) \quad \frac{1}{\sqrt{8\pi}} \int |e^{\gamma u - \gamma^2/2} - 1| e^{-u^2/2} du.$$

The conclusion then follows by performing a few changes of variables.

PROOF OF LEMMA 3.1. Statement (ii) follows from a more general statement on concentration for Lipschitz functional on  $\Omega_N$ : Proposition B.1 is proved in the [Appendix](#).

To ensure that the convergence holds uniformly in  $\theta$ , the reader can check that all the bounds present in the proof do not depend on  $\theta$ . In the remainder of the paper, we will use the notation

$$(3.11) \quad \overline{\sin}(x) = \overline{\sin}_\theta(x) := \sin\left(\frac{x\pi}{N} + \theta\right).$$



As most computations do not depend on the value of  $\theta$ , we omit it from the notation most of the time in the remainder of the paper. Note that  $a(\eta)$  satisfies trivially  $\mu_N(a(\eta)) = 0$ . Let us show that the variance is asymptotically equivalent to  $N$ .

$$(3.12) \quad \mu_N(a(\eta)^2) = \sum_{x \in \mathbb{Z}_{2N}} \overline{\sin}(x)^2 + \sum_{\substack{(x,y) \in \mathbb{Z}_{2N} \\ x \neq y}} \overline{\sin}(x) \overline{\sin}(y) \mathbb{E}[\eta(x)\eta(y)].$$

The first term is equal to  $N$ . As for the second term, as we have  $\mathbb{E}[\eta(x)\eta(y)] = -1/(2N+1)$  it is equal to

$$(3.13) \quad \frac{1}{(2N-1)} \sum_{x \in \mathbb{Z}_{2N}} \overline{\sin}(x)^2 = \frac{N}{2N-1}.$$

To show the convergence to a Gaussian variable, we will use the martingale central limit theorem [2], Theorem 1. Let  $(M_i^N)_{i=0}^{2N-1}$  be the martingale defined by

$$(3.14) \quad M_i^N := \mu_N(a(\eta) | (\eta(x))_{x=1}^i).$$

It satisfies  $M_0^N = 0$  and  $M_{2N-1}^N = a(\eta)$ . Set

$$(3.15) \quad \Delta M_i := M_{i+1}^N - M_i^N$$

and

$$(3.16) \quad \sigma_{i,N}^2 = \mu_N((\Delta M_i)^2 | (\eta(x))_{x=1}^i).$$

To apply the central limit theorem the martingale  $M_i$ , one must only check that

$$(3.17) \quad \sigma_N^2 := \frac{1}{N} \sum_{i=0}^{2N-2} \sigma_{i,N}^2,$$

converges to one, in probability (there are in fact other assumptions to check; see [2] but in our case they are trivially satisfied).

For  $A \subset \mathbb{Z}_{2N}$ , we let  $\eta(A)$  denote the number of particles in the set  $A$ ,

$$(3.18) \quad \eta(A) := \sum_{x \in A} \mathbf{1}_{\{\eta_x=1\}}.$$

Let us fix  $i$  and set  $k = k(\eta, i) := \eta([1, i])$ . A simple computation gives

$$(3.19) \quad \Delta M_i = \begin{cases} \frac{N+k-i}{2N-i} \left( \overline{\sin}(i+1) - \frac{1}{2N-i-1} \sum_{x=i+2}^{2N} \overline{\sin}(x) \right), & \text{if } \eta(i+1) = 1, \\ \frac{N-k}{2N-i} \left( \overline{\sin}(i+1) - \frac{1}{2N-i-1} \sum_{x=i+2}^{2N} \overline{\sin}(x) \right), & \text{if } \eta(i+1) = -1. \end{cases}$$

As the first and second option in (3.18) have respective probability  $(N - k)/(2N - i)$  and  $(N + k - i)/(2N - i)$ , we have

$$(3.20) \quad \sigma_{i,N}^2 = \frac{2(N + k - i)(N - k)}{(2N - i)^2} \left( \overline{\sin}(i + 1) - \frac{1}{2N - i - 1} \sum_{x=i+2}^{2N} \overline{\sin}(x) \right)^2.$$

Once this is done, we just need to check the following facts to conclude:

- (a) For all  $i$ ,  $\sigma_{i,N}^2$  is almost surely smaller than 8.
- (b) For all  $N$  sufficiently large, for all  $i \in [0, 2N - \sqrt{N}]$  we have

$$(3.21) \quad \mu_N \left( \left| \sigma_{i,N}^2 - \frac{1}{2} \left( \overline{\sin}(i + 1) - \frac{1}{2N - i - 1} \sum_{x=i+2}^{2N} \overline{\sin}(x) \right)^2 \right| \geq N^{-1/20} \right) \leq N^{-1/20}.$$

- (c) We have the following convergence:

$$(3.22) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^{2N-1} \left( \overline{\sin}(i + 1) - \frac{1}{2N - i - 1} \sum_{x=i+2}^{2N} \overline{\sin}(x) \right)^2 = 2.$$

From these three claims, it is rather standard to show that  $\sigma_N^2$  converges to 1 in probability and we leave it as an exercise to the reader. Item (a) is obvious, item (b) follows from computing the mean and variance of  $k(\eta, i)$  [which are, resp., equal to  $i/2$  and  $i(2N - i)/4(2N - 1)$ ] and applying the Markov inequality. As for (c), it can be checked via a tedious but simple computation.  $\square$

**4. Decomposing the proof of Proposition 2.3.** We present in this section the main steps of the proof of our main result and the heuristics behind it.

4.1. *Why coupling with  $\nu^{N,\alpha,\theta}$ ?* First, let us try to understand why  $\nu^{N,\alpha,\theta}$  gives a good approximation of the  $P_t^X$ , via an analysis of the particle density and fluctuation. Let

$$(4.1) \quad u^X(x, t) := \mathbb{E}[\eta_t^X(x)]$$

denote the expected particle density (with our convention it can be negative since empty sites count for  $-1$ ). It is standard to check that  $u^X$  is the solution of the discrete heat-equation

$$(4.2) \quad \begin{cases} \partial_t u(x, t) := \Delta u(x, t), \\ u(x, 0) := \chi(x), \end{cases}$$

where  $\Delta$  denotes the discrete Laplacian

$$(4.3) \quad \Delta u(x, t) := u(x + 1, t) + u(x - 1, t) - 2u(x, t).$$

Here and in what follows if  $f$  is a function of  $\mathbb{Z}_{2N}$  (identified to a periodic function of  $\mathbb{Z}$ ) such that

$$(4.4) \quad \sum_{x \in \mathbb{Z}_{2N}} f(x) = 0,$$

and  $x$  and  $y$  are two elements of  $\mathbb{Z}_{2N}$  and  $x_0 \leq y_0$  two elements of  $\mathbb{Z}$  which are, respectively, equal to  $x$  and  $y$  modulo  $2N$ . Then we use the notation  $\sum_{z=x}^y f(z)$ , to denote the sum  $\sum_{z=x_0}^{y_0} f(z)$ . It is straightforward that it does not depend on the particular choice of  $x_0$  and  $y_0$  once  $x$  and  $y$  are fixed. Let us quickly investigate the fluctuations of the integrated density of particle

$$(4.5) \quad \xi(\eta)(x) := \sum_{z=1}^x \eta(z).$$

At equilibrium,  $\xi(\eta)$  is a simple random-walk conditioned to return to zero after  $2N$  steps. It is centered and has Gaussian fluctuations of order  $\sqrt{N}$ . In [9], we have proved that the fluctuation of  $\xi(\eta_t^\chi)(x)$  around its mean [given by  $\sum_{z=1}^x u^\chi(z, t)$ ] are in fact always of order  $\sqrt{N}$ .

This gives the intuition that much before the mixing time, the law of  $\eta_t^\chi$  might approximately be that of  $2N$  independent  $\pm 1$  Bernoulli variables, each with bias  $u^\chi(x, t)$ , conditioned to  $\sum_{x \in \mathbb{Z}_{2N}} \eta_t^\chi(x) = 0$ .

For  $t \geq \frac{N^2}{4\pi^2} \log N$ ,  $u^\chi(x)$  is very well approximated by a sinusoid function (see Lemma 5.3)

$$(4.6) \quad u^\chi(x, t) \approx b(\chi) e^{-\lambda_N t} \sin\left(\frac{\pi x}{N} + \theta(\chi)\right),$$

and the conditioned law of independent Bernoulli described above is very close in total variation to  $\nu^{N, b(\chi) e^{-\lambda_N t}, \theta(\chi)}$ .

**4.2. How to do it.** Let us first write here the rigorous result concerning the fluctuation around the expected density of particle.

**PROPOSITION 4.1.** *There exists a constant  $c > 0$  such that for all  $N$  sufficiently large, for all  $t \geq \frac{1}{4}(\lambda_N)^{-1} \log N$ , we have*

$$(4.7) \quad \begin{aligned} P_t^\chi \left[ \exists x, y \in \mathbb{Z}_{2N}, \left| \sum_{z=x+1}^y \left[ \eta(z) - e^{-\lambda_N t} b(\chi) \sin\left(\frac{\pi z}{N} + \theta(\chi)\right) \right] \right| \geq s\sqrt{N} \right] \\ \leq 2e^{-cs^2} \end{aligned}$$

In particular, we know that with high probability,  $\eta_t^\chi$  lies in the set

$$(4.8) \quad \mathcal{G}_{\alpha,\theta}^N := \left\{ \eta \in \Omega_N \left| \max_{x,y \in \mathbb{Z}_{2N}} \left| \sum_{z=x+1}^y \left( \eta(z) - \alpha \sin \left( \frac{\pi z}{N} + \theta \right) \right) \right| \leq \sqrt{N} \log \log N \right. \right\}$$

with  $\alpha$  and  $\theta$  being chosen, respectively, equal to  $e^{-\lambda_N t} b(\chi)$  and  $\theta(\chi)$ .

To prove Proposition 2.3, it is sufficient to prove that:

- within a time  $N^2(\log N)^{1/2}$  (i.e., a shorter time-scale than the mixing time), one can couple a dynamics with initial condition  $\chi \in \mathcal{G}_{\alpha,\theta}^N$ , with one with initial condition distributed like  $\nu^{N,\alpha,\theta}$ .
- the family of measure  $(\nu^{N,\alpha,\theta})$  is almost preserved by the dynamics in the sense that applying the semi-group  $P_t$  to it only changes the value of  $\alpha$ .

Both of these statements hold provided  $\alpha$  is sufficiently small, and are stated as two propositions below. More precisely Let  $\nu_t^{N,\alpha,\theta}$  be the law of a system started with initial configuration  $\nu^{N,\alpha,\theta}$

$$(4.9) \quad \nu_t^{N,\alpha,\theta}(\eta) := \sum_{\eta' \in \Omega_N} \nu_t^{N,\alpha,\theta} P_t(\eta', \eta).$$

**PROPOSITION 4.2.** *For all  $N$  sufficiently large, for all  $\theta \in [0, 2\pi)$ , for all  $\alpha \leq 2N^{-3/7}$ , we have for all  $\chi \in \mathcal{G}_{\alpha,\theta}$ , for all  $t \geq N^2(\log N)^{1/2}$*

$$(4.10) \quad \|P_t^\chi - \nu_t^{N,\alpha,\theta}\| \leq \frac{1}{2 \log \log N}.$$

**PROPOSITION 4.3.** *There exists a constant  $C$  such that for all  $N$  and all  $\alpha > 0$ ,*

$$(4.11) \quad \sup_{t \geq 0} \|\nu_t^{N,\alpha,\theta} - \nu^{N,\alpha e^{-\lambda_N t}, \theta}\|_{\text{TV}} \leq C \alpha^2 \sqrt{N}.$$

**PROOF OF PROPOSITION 2.3.** We have for any  $\chi$  in  $\beta$  for  $t \geq t_0 := 3/7(\lambda_N)^{-1}$  we have, using the triangular inequality

$$(4.12) \quad \begin{aligned} & \|P_t^\chi - \nu^{N,b(\chi)e^{-\lambda_N t}, \theta(\chi)}\|_{\text{TV}} \\ & \leq \sum_{\chi' \in \Omega_N} P_t(\chi, \chi') \|P_{t-t_0}^{\chi'} - \nu^{N,b(\chi)e^{-\lambda_N t}, \theta(\chi)}\|_{\text{TV}} \\ & \leq P_{t_0}^\chi(\eta \notin \mathcal{G}_{b(\chi)N^{-3/7}, \theta(\chi)}^N) \\ & \quad + \max_{\chi' \in \mathcal{G}_{b(\chi)N^{-3/7}, \theta(\chi)}^N} \|P_{t-t_0}^{\chi'} - \nu^{N,b(\chi)e^{-\lambda_N t}, \theta(\chi)}\|_{\text{TV}}. \end{aligned}$$

According to Proposition 4.1, we have

$$(4.13) \quad P_{t_0}^\chi(\eta \notin \mathcal{G}_{b(\chi)N^{-3/7}, \theta(\chi)}^N) \leq \frac{1}{\log N}.$$

We note that for  $\chi' \in \mathcal{G}_{b(\chi)N^{-3/7}, \theta(\chi)}^N$  we have

$$(4.14) \quad \begin{aligned} & \|P_{t-t_0}^{\chi'} - \nu^{N, b(\chi)N^{-3/7}, \theta(\chi)}\|_{\text{TV}} \\ & \leq \|P_{t-t_0}^{\chi'} - \nu_{t-t_0}^{N, b(\chi)N^{-3/7}, \theta(\chi)}\|_{\text{TV}} \\ & \quad + \|\nu_{t-t_0}^{N, b(\chi)N^{-3/7}, \theta(\chi)} - \nu^{N, b(\chi)N^{-3/7}, \theta(\chi)}\|_{\text{TV}}. \end{aligned}$$

The first term is smaller than  $\frac{1}{2 \log \log N}$  according to Proposition 4.2 as soon as

$$t \geq t_0 + N^2 \sqrt{\log N}.$$

Proposition 4.3 ensures that the second term is smaller than  $(\log N)^{-1}$ , hence we can conclude.  $\square$

**5. Proof of Proposition 4.1.** This statement is in fact mostly derived from the statement about fluctuations proved in [9] which we state now.

PROPOSITION 5.1 ([9], Proposition 4.1). *There exists a constant  $c > 0$  such that for all  $t \geq 0$ , for all  $s \geq 0$ , for all  $\chi \in \Omega_N$  we have*

$$(5.1) \quad P_t^\chi \left[ \exists x, y \in \mathbb{Z}_{2N}, \left| \sum_{z=x+1}^y (\eta(z) - u^\chi(z, t)) \right| \geq s\sqrt{N} \right] \leq 2e^{-cs^2}.$$

REMARK 5.2. Note that in [9],  $t \geq 3N^2$  is required (that would be in fact  $t \geq 12N^2$  in our setup because we work on  $\mathbb{Z}_{2N}$  instead of  $\mathbb{Z}_N$ ), but this is only to treat the case of an arbitrary number of particles. The reader can check from the proof that this assumption is only needed to check [9], equation (4.4), which is obviously valid for all  $t \geq 0$  when we have  $N$  particles on  $2N$  sites.

With this result, Proposition 4.1 follows from a basic analysis of the Fourier coefficients of the solution of (4.2).

LEMMA 5.3. *For all  $t \geq \frac{1}{4}(\lambda_N)^{-1} \log N$ , we have*

$$(5.2) \quad \max_{x \in \mathbb{Z}_{2N}} \left| u^\chi(x, t) - e^{-\lambda_N t} b(\chi) \sin \left( \frac{\pi x}{N} + \theta(\chi) \right) \right| \leq 4N^{-1/2}.$$

PROOF. Let us decompose  $u^\chi$  along its Fourier modes for the heat-equation.

As in Section 2.3, we prefer to have only one coefficient per eigenspace, and thus, for  $i = 2, \dots, N-1$ , introduce  $\theta_i(\chi)$  to be the phase of the projection. We let  $\theta_i(\chi)$  be either the unique solution of

$$(5.3) \quad \begin{aligned} \sum_{x \in \mathbb{Z}_{2N}} \chi(x) \cos\left(\frac{i\pi x}{N} + \theta\right) &= 0, \\ \sum_{x \in \mathbb{Z}_{2N}} \chi(x) \sin\left(\frac{i\pi x}{N} + \theta\right) &> 0 \end{aligned}$$

or 0 if all  $\theta$  are solution, and let us set

$$(5.4) \quad \begin{aligned} b_i(\chi) &:= \frac{1}{N} \sum_{z \in \mathbb{Z}_{2N}} \chi(z) \sin\left(\frac{i\pi z}{N} + \theta_i(\chi)\right), \\ b_N(\chi) &:= \frac{1}{2N} \sum_{z \in \mathbb{Z}_{2N}} (-1)^{|z|} \chi(z). \end{aligned}$$

We have

$$(5.5) \quad \chi(x) := \sum_{i=1}^{N-1} b_i(\chi) \sin\left(\frac{i\pi x}{N} + \theta_i(\chi)\right) + b_N(\chi) (-1)^{|x|}.$$

As the functions  $x \mapsto \sin(\frac{i\pi x}{N} + \theta_i)$  are eigenfunctions of the discrete Laplacian with respective eigenvalues,

$$(5.6) \quad -\lambda_{i,N} := 2\left(1 - \cos\left(\frac{i\pi}{N}\right)\right),$$

we have for all  $t \geq 0$ ,

$$(5.7) \quad u(x, t) = e^{-\lambda_N t} b(\chi) \sin\left(\frac{\pi x}{N} + \theta(\chi)\right) + R(\chi, t, x),$$

where

$$(5.8) \quad R(\chi, t, x) := \sum_{i=2}^{N-1} e^{-\lambda_{i,N} t} b_i(\chi) \sin\left(\frac{i\pi x}{N} + \theta_i(\chi)\right) + b_N(\chi) e^{-2t}.$$

Noticing that  $|b_i(\chi)| \leq 2$  and that for all  $N \geq 2$ ,

$$(5.9) \quad \forall i \in \{2, \dots, N\}, \quad \lambda_{i,N} \geq i\lambda_N,$$

we have for all  $t \geq (\lambda_N)^{-1}$

$$(5.10) \quad |R(\chi, t, x)| \leq 2 \sum_{i=2}^N e^{-i\lambda_N t} = \frac{2e^{-2\lambda_N t}}{1 - e^{-\lambda_N t}} \leq 4e^{-2\lambda_N t}.$$

Hence, we have the result.  $\square$

**6. Proof of Proposition 4.2.** We assume without loss of generality that  $\alpha$  is nonnegative, and write  $\nu^\alpha$  and  $\nu_t^\alpha$  for  $\nu^{N,\alpha,0}$  and  $\nu_t^{N,\alpha,0}$ , and  $a(\eta)$  for  $a_0(\eta)$ .

6.1. *Properties of  $\nu^\alpha$ .* In this section, we check several properties for  $\nu^\alpha$ . While the results are quite intuitive, their proof is quite technical and we have decided to postpone them to Appendix A. First, we want to ensure that it has the right density of the particle.

PROPOSITION 6.1. *There exists a constant  $C$  such that for all  $\alpha \leq 1$  we have*

$$(6.1) \quad \sup_{x \in \mathbb{Z}_{2N}} |\nu^\alpha(\eta(x)) - \alpha \overline{\sin}(x)| \leq C(\alpha^2 + N^{-2}).$$

Then we have to check that the fluctuations are not larger than  $\sqrt{N}$ .

PROPOSITION 6.2. *There exists constant  $c$  such that for all  $N > 0$ , for all  $|\alpha| \leq N^{-1/4}$ , and  $t \geq 0$*

$$(6.2) \quad \nu_t^\alpha \left[ \exists x, y \in \mathbb{Z}_{2N}, \left| \sum_{z=x+1}^y (\eta(z) - \alpha e^{-\lambda_N t} \overline{\sin}(z)) \right| \geq s\sqrt{N} \right] \leq 2e^{-cs^2}.$$

Finally, we want to check that if one starts from distribution  $\nu^\alpha$  there is a positive density of sites where  $\eta(z) \neq \eta(z+1)$ , that is, of locations where jumps of the particle can occur. The utility of such a statement will be become clear in the next section when we construct the dynamical coupling. For a probability measure  $\nu$  defined on  $\Omega_N$ , we let  $\mathbb{P}^\nu$  be the law of the Markov chain  $(\eta_t)_{t \geq 0}$  starting from  $\eta_0$  distributed like  $\nu$ . Set

$$(6.3) \quad j(x, y, \eta) := \{z \in [x, y] | \eta(z) \neq \eta(z+1)\}$$

and

$$(6.4) \quad \mathcal{E} := \{\eta \in \Omega_N | \forall (x, y) \in \mathbb{Z}_{2N}^2, \# [x, y] \geq N^{1/4} \Rightarrow j(x, y, \eta) \geq \frac{1}{4} \# [x, y]\}.$$

PROPOSITION 6.3. *There exist a constant  $c$  such that for  $N$  sufficiently large, for all  $|\alpha| \leq cN^{-3/8}$*

$$(6.5) \quad \mathbb{P}^{\nu^\alpha} [\exists t \leq N^3, \eta_t \notin \mathcal{E}] \leq e^{-cN^{1/4}}.$$

REMARK 6.4. The power exponents for  $N$  in Proposition 6.3 are rather arbitrary and other choices would also fit. The important result is that the probability tends to zero.

6.2. *The  $\xi$  dynamics.* We introduce in this section an auxiliary dynamics (the same as in [9]) which is used to couple  $P_t^\chi$  with  $\chi \in \mathcal{G}_\alpha$  (we use this notation for  $\mathcal{G}_{\alpha,0}^N$ ) with  $\nu_t^\alpha$ . The idea of using interface dynamics to study particle system dates is not new and is already present in the seminal paper of Rost about the asymmetric exclusion on the line [18] (for the use of this technique for mixing time related issues, see [8, 9, 19]). In [8, 19], the height function is introduced mainly to have a better intuition on an order which can be defined without the interface representation. Let us stress that here, on the contrary, the interface dynamics is used to perform a monotone coupling that could not be constructed by considering only the original chain.

Let us consider the set of discrete height functions of the circle.

$$(6.6) \quad \Omega'_N := \{\xi : \mathbb{Z}_{2N} \rightarrow \mathbb{Z} \mid \xi(0) \in 2\mathbb{Z}, \forall x \in \mathbb{Z}_{2N}, |\xi(x) - \xi(x+1)| = 1\}.$$

Given  $\xi$  in  $\Omega'_N$ , we define  $\xi^x$  as

$$(6.7) \quad \begin{cases} \xi^x(y) = \xi(y), & \forall y \neq x, \\ \xi^x(x) = \xi(x+1) + \xi(x-1) - 2\xi(x). \end{cases}$$

We let  $\xi_t$  be the irreducible Markov chain on  $\Omega'_N$  whose transition rates  $p$  are given by

$$(6.8) \quad \begin{cases} p(\xi, \xi^x) = 1, & \forall x \in \mathbb{Z}_N, \\ p(\xi, \xi') = 0, & \text{if } \xi' \notin \{\xi^x \mid x \in \mathbb{Z}_N\}. \end{cases}$$

We call this dynamics the corner-flip dynamics, as the transition  $\xi \rightarrow \xi^x$  corresponds to flipping either a local maximum of  $\xi$  (a “corner” for the graph of  $\xi$ ) to a local minimum or vice versa. It is of course not positive recurrent, as the state space is infinite and translation invariant for the dynamics, however, it is irreducible and recurrent.

The reader can check that  $\Omega'_N$  is mapped onto  $\Omega_N$ , by the transformation  $\xi \mapsto \nabla \xi$  where

$$(6.9) \quad \nabla \xi(x) := \xi(x+1) - \xi(x)$$

and that the image of the corner-flip dynamics  $(\nabla \xi_t)_{t \geq 0}$  is the simple exclusion process (see Figure 1).

There is a natural order on the set  $\Omega'_N$  defined by

$$(6.10) \quad \xi \geq \xi' \iff \forall x \in \mathbb{Z}_{2N}, \quad \xi(x) \geq \xi'(x),$$

and we can construct a *grand coupling* for the Markov chain which preserves this order.



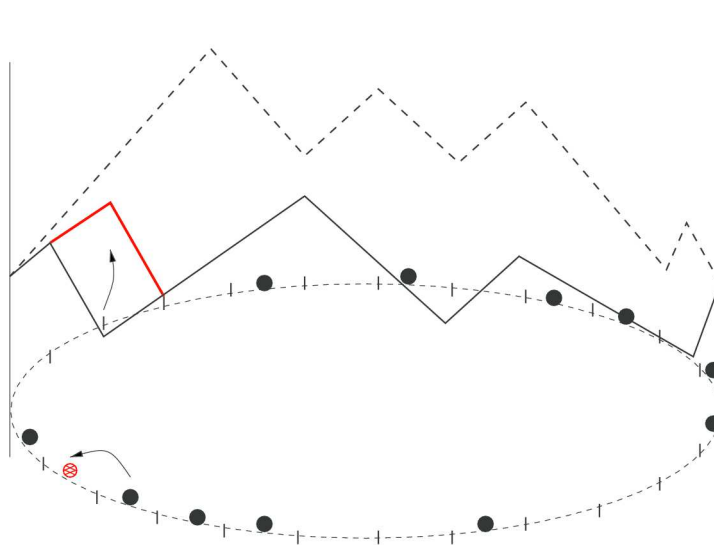


FIG. 1. *The correspondence between the exclusion process and the corner-flip dynamics. A particle jump and its corner-flip counterpart are indicated by arrows. Note that this is not a one-to-one mapping as a particle configuration gives the height function only modulo translation.*

**6.3. The graphical construction.** We introduce in this section an order preserving grand-coupling on  $\Omega'_N$ . For  $\zeta \in \Omega'_N$ ,  $(\xi_t^\zeta)_{t \geq 0}$  denotes the Markov chain with initial condition  $\zeta$ . We want to construct all the  $(\xi_t^\zeta)_{t \geq 0}$  on a same probability space in a way that

$$(6.11) \quad \forall \zeta, \zeta' \in \Omega'_N, (\zeta \geq \zeta') \Rightarrow (\forall t \geq 0, \xi_t^\zeta \geq \xi_t^{\zeta'}).$$

Of course, there are several options for such a grand coupling. We want to choose one which is such that, eventually, the trajectories starting from different initial conditions coalesce almost surely (at a random time)

$$(6.12) \quad \forall \zeta, \zeta' \in \Omega'_N, \exists T_{\zeta, \zeta'} < \infty, \forall t \geq T_{\zeta, \zeta'}, \quad \xi_t^\zeta = \xi_t^{\zeta'}.$$

Of course, we want the coalescing time to be as short as possible. To reach this aim, we make the different corner flips for different trajectories as independent as can be while still satisfying (6.11).

Let us present the construction. The evolution of the  $(\xi_t)_{t \geq 0}$  is completely determined by auxiliary Poisson processes which we call clock processes. Set

$$\Theta := \{(x, z) | x \in \mathbb{Z}_N \text{ and } z \in 2\mathbb{Z} + (-1)^x\}.$$

And set  $\mathcal{T}^\uparrow$  and  $\mathcal{T}^\downarrow$  to be two independent rate-one clock processes indexed by  $\Theta$  ( $\mathcal{T}_\omega^\uparrow$  and  $\mathcal{T}_\omega^\downarrow$  are two independent Poisson processes of intensity one of

each  $\omega \in \Theta$ ). The trajectory of  $\xi_t$  given  $(\mathcal{T}^\uparrow, \mathcal{T}^\downarrow)$  is given by the following construction:

- $\xi_t$  is a càdlàg, and does not jump until one of the clocks indexed by  $(x, \xi_t(x))$ ,  $x \in \mathbb{Z}_{2N}$  rings.
- If  $\mathcal{T}_{(x, \xi_{t-}(x))}^\downarrow$  rings at time  $t$  and  $x$  is a local maximum for  $\xi_{t-}$ , then  $\xi_t = \xi_{t-}^x$ .
- If  $\mathcal{T}_{(x, \xi_{t-}(x))}^\uparrow$  rings at time  $t$  and  $x$  is a local minimum for  $\xi_{t-}$ , then  $\xi_t = \xi_{t-}^x$ .

6.4. *Construction the initial condition for  $\xi^0$ ,  $\xi^1$  and  $\xi^2$ .* Given  $\chi \in \mathcal{G}_\alpha$ , we let  $(\xi_t^0)$  the trajectory of the Markov chain with transitions rates (6.8) starting from initial condition

$$(6.13) \quad \xi_0^0(x) := \sum_{z=0}^x \chi(x).$$

Note that for all  $t \geq 0$  we have

$$(6.14) \quad \mathbb{P}[\nabla \xi_t^0 \in \cdot] = P_t^\chi.$$

Our idea is to construct another dynamic  $\xi_t^1$  which starts with  $\nabla \xi_0^1$  distributed like  $\nu^\alpha$  which coalesces with  $\xi_t^0$  within time  $N^2 \sqrt{\log N}$ . In fact, it turns out more practical to define not one but two dynamics  $\xi^1$  and  $\xi^2$  to couple with  $\xi^0$ . We let  $\mathbb{P}$  denote the law of  $(\xi_t^0, \xi_t^1, \xi_t^2)_{t \geq 0}$ , and we impose

$$(6.15) \quad \mathbb{P}[\nabla \xi_0^1 \in \cdot] = \mathbb{P}[\nabla \xi_0^2 \in \cdot] = \nu^\alpha.$$

Note that this implies for all  $t \geq 0$

$$(6.16) \quad \mathbb{P}[\nabla \xi_t^1 \in \cdot] = \mathbb{P}[\nabla \xi_t^2 \in \cdot] = \nu_t^\alpha.$$

We impose also the condition

$$(6.17) \quad \xi_0^1 \leq \xi_0^0 \leq \xi_0^2,$$

and use the graphical coupling introduced in the previous section to construct the trajectory of  $(\xi_t^i)_{t \geq 0}$ ,  $i = 0, 1, 2$ . Hence, the order is conserved at all time

$$(6.18) \quad \forall t \geq 0, \quad \xi_t^1 \leq \xi_t^0 \leq \xi_t^2.$$

Let us now explain our construction of the initial conditions. We start with  $\eta_0$  distributed like  $\nu^\alpha$  and we will choose  $\xi_0^1$  and  $\xi_0^2$  such that

$$(6.19) \quad \nabla \xi_0^1 = \nabla \xi_0^2 = \eta_0.$$

We set for arbitrary  $\eta \in \Omega_N$ , or  $\xi \in \Omega'_N$

$$(6.20) \quad H_{t,\alpha}(\eta) := \max_{x,y \in \mathbb{Z}_N} \left| \sum_{z=x+1}^y \eta(z) - e^{-\lambda_N t} \overline{\sin}(t) \right|,$$

$$H_{t,\alpha}(\xi) := H_{t,\alpha}(\nabla \xi).$$

We also set

$$(6.21) \quad \mathcal{H}_0 := 2[(H_{0,\alpha}(\eta_0) + \sqrt{N} \log \log N)/2]$$

and

$$(6.22) \quad \begin{aligned} \xi_0^1(x) &:= \sum_{z=1}^x \eta_0(z) - \mathcal{H}_0, \\ \xi_0^2(x) &:= \sum_{z=1}^x \eta_0(z) + \mathcal{H}_0. \end{aligned}$$

The fact that (6.17) is satisfied follows from the definition of  $\mathcal{G}_\alpha$  and that of  $\mathcal{H}_0$ . Note also that from Proposition 6.2 applied at  $t = 0$ , we have

$$(6.23) \quad \mathbb{P}[\mathcal{H}_0 \geq 2\sqrt{N} \log \log N] \leq (\log N)^{-1}.$$

To prove Proposition 4.2, it is sufficient to prove that  $\xi_t^1$  and  $\xi_t^2$  typically coalesce within a time  $N^2 \sqrt{\log N}$ . More precisely, we have the following.

**PROPOSITION 6.5.** *For sufficiently large  $N$ , for all  $\alpha \leq 2N^{-3/7}$  for  $(\xi_t^1)_{t \leq 0}$ ,  $(\xi_t^2)_{t \geq 0}$ , constructed as above, we have*

$$(6.24) \quad \mathbb{P}[\xi_{N^2 \sqrt{\log N}}^1 \neq \xi_{N^2 \sqrt{\log N}}^2] \leq \frac{1}{2 \log \log N}.$$

Proposition 6.5 is proved in Sections 6.5 and 6.6.

**PROOF OF PROPOSITION 4.2.** Let  $\chi$  in  $\mathcal{G}_\alpha$  be fixed and consider the dynamics  $\xi^i$ ,  $i = 0, 1, 2$  constructed as above. From (6.18), we have

$$(6.25) \quad \xi_t^1 = \xi_t^2 \quad \Rightarrow \quad \xi_t^1 = \xi_t^0.$$

Recalling (6.14) and (6.16), we have for any  $t > 0$

$$(6.26) \quad \|P_t^\chi - \nu_t^\alpha\|_{\text{TV}} \leq \mathbb{P}[\nabla \xi_t^0 \neq \nabla \xi_t^1] \leq \mathbb{P}[\xi_t^0 \neq \xi_t^1] \leq \mathbb{P}[\xi_t^1 \neq \xi_t^2].$$

Hence, Proposition 6.5 implies the result.  $\square$

**6.5. The randomly walking area.** Let us set

$$(6.27) \quad A(t) = \frac{1}{2} \sum_{x \in \mathbb{Z}_2} \xi_t^2(x) - \xi_t^1(x).$$

The reader can check that  $A(t)$  is an integer. Because of (6.18), we remark that  $A(t)$  is always positive, and hence that  $\xi^1$  and  $\xi^2$  merge at time

$$(6.28) \quad \tau := \inf\{t \geq 0 | A(t) = 0\}.$$

As  $A(t)$  is an integer valued martingale which only makes  $\pm 1$  jumps, it is to be a time changed symmetric nearest neighbor walk on  $\mathbb{Z}_+$ . In order to get a bound for

$$\mathbb{P}[\tau \leq t],$$

we need to have a reasonable control over the time change, that is, the jump rate of  $A(t)$ . It depends on the particular configuration  $(\xi_t^1, \xi_t^2)$  the system sits on: it is given by the number of places where corners can flip independently for  $\xi_t^1$  and  $\xi_t^2$ . More precisely, set

$$(6.29) \quad \begin{aligned} U_i(t) &:= \{x \in \mathbb{Z}_N \mid \xi_t^i \text{ has a local extremum at } x \text{ and} \\ &\quad \exists y \in \{x-1, x, x+1\}, \xi_t^2(y) > \xi_t^1(y)\}. \end{aligned}$$

The jump rate of  $A(t)$  is given by

$$(6.30) \quad u(t) := \#U_1(t) + \#U_2(t).$$

For  $t \leq \int_0^\tau u(t) dt$ , let us define

$$(6.31) \quad J(t) := \inf \left\{ s \mid \int_0^s u(v) dv \geq t \right\}.$$

By construction, the process  $(X_t)_{t \geq 0}$  defined by

$$(6.32) \quad X_t := A(J(t))$$

is a continuous time random walk on  $\mathbb{Z}_+$  which jumps up and down with rate  $1/2$ . From the definition, we have

$$(6.33) \quad X_0 = A(0) := N\mathcal{H}_0.$$

Note that from Proposition 6.2, and the definition of  $\mathcal{H}_0$  we have

$$(6.34) \quad \mathbb{P}[A(0) \geq 2N^{3/2} \log \log N] \leq (\log N)^{-1}.$$

To estimate  $\tau$ , we have to control the evolution of  $X_t$  (using standard properties of the random walk) and that of  $u(t)$  (using the properties of proved in Section 6.1).

**6.6. Multiscale analysis.** To have the best possible control on  $u(t)$ , we need to perform a multi-scale analysis. We construct a sequence of intermediate stopping time  $(\tau_i)_{i \geq 0}$  as follows:

$$(6.35) \quad \tau_i := \inf\{t \geq 0 \mid A(t) \leq N^{3/2} 2^{-i}\}.$$

We set  $\tau_{-1} := 0$  for convenience. We are interested in  $\tau_i$  for  $i \in \{0, \dots, \lceil (\log_2 N)/2 \rceil\}$  where  $\log_2(\cdot) := \log(\cdot)/\log(2)$  denotes the logarithm in base 2. To bound the value of  $\tau$ , we bound the value of each  $\Delta\tau_i = \tau_i - \tau_{i-1}$  for  $i \leq \lceil (\log_2 N)/2 \rceil$  and that of  $\tau - \tau_{\lceil (\log_2 N)/2 \rceil}$ . The way to do this is:

(i) First, we prove a bound for the analog of the  $\Delta\tau_i$  for the process  $X_t$  defined in (6.32).

(ii) Second, we prove a bound for  $u(t)$  which is valid in the interval  $[\tau_{i-1}, \tau_i]$ .

For step (i), let us define

$$(6.36) \quad \begin{aligned} \mathcal{T}_i &:= \int_{\tau_{i-1}}^{\tau_i} u(t) dt, \\ \mathcal{T}_\infty &:= \int_{\tau_{\lceil (\log_2 N)/2 \rceil}}^{\tau} u(t) dt. \end{aligned}$$

It follows from standard properties of the random walk and from (6.34) that we have the following.

LEMMA 6.6. *We have the following estimates:*

$$(6.37) \quad \begin{aligned} \mathbb{P}[\exists i \in \{0, \dots, \lceil (\log_2 N)/2 \rceil\}, \mathcal{T}_i \geq 3^{-i} N^3 (\log N)^{1/4}] &\leq (\log N)^{-1/10}, \\ \mathbb{P}[\mathcal{T}_\infty \geq N^2 (\log N)^{1/4}] &\leq (\log N)^{-1/10}. \end{aligned}$$

For more details, we refer to the proof of [9], Lemma 6.1.

Step (ii) is more delicate, because we cannot get a good bound on  $u$  which is uniform in time. For instance, we need to prove that most of the time  $u(t)$  is of order  $N$  but we know that just before  $\tau$  we have  $u(t) = 4$ . Hence, we will prove a different bound for each value of  $i$ . The bound is valid most of the time, and we will need to check that the small fraction of time during which it does not hold can be dealt with in the computations. Recalling (6.20), we set

$$(6.38) \quad \mathcal{H}(t) := \max(H_{\alpha,t}(\xi_t^1) + H_{\alpha,t}(\xi_t^2), \sqrt{N}).$$

We notice that from the definition

$$(6.39) \quad \max_{x \in \mathbb{Z}} (\xi_t^2(x) - \xi_t^1(x)) \leq \mathcal{H}(t).$$

Using this information, we can get the following control on  $u$  [recall (6.4)]:

LEMMA 6.7. *If  $\xi_t^1 \in \mathcal{E}$ , we have*

$$(6.40) \quad u(t) \geq \frac{1}{8} \min\left(N, \frac{A(t)}{\mathcal{H}(t)}\right).$$

The proof is identical to the one of [9], Lemma 6.3. Note that thanks to Proposition (6.3) and our assumption  $\alpha \leq 2N^{-3/7}$ , the inequality (6.40) is valid up to time  $N^3$  (which is much more than what we need) with high probability. To make this bound on  $u$  useful, we need to show that most of the time  $\mathcal{H}(t)$  is not too large.

LEMMA 6.8. *For any  $T \geq 0$ ,*

$$(6.41) \quad \mathbb{P} \left[ \int_0^T \mathbf{1}_{\{\mathcal{H}(t) \geq \sqrt{N} \log \log N\}} dt \geq T(\log N)^{-4} \right] \leq (\log N)^{-1}.$$

PROOF. It follows from (6.2) that for  $N$  sufficiently large, for any  $t \geq 0$

$$(6.42) \quad \mathbb{P}[\mathcal{H}(t) \geq \sqrt{N} \log \log N] \leq (\log N)^{-5}.$$

Then the result follows by using the Markov property for the integrated inequality.  $\square$

PROOF OF PROPOSITION 6.5. Set

$$(6.43) \quad \begin{aligned} \mathcal{A} &:= \{\forall t \leq N^3, \xi_t^1 \in \mathcal{E}\}, \\ \mathcal{B} &:= \left\{ \int_0^T \mathbf{1}_{\{\mathcal{H}(t) \geq \sqrt{N} \log \log N\}} dt \leq T(\log N)^{-5} \right\}, \\ \mathcal{C} &:= \{\mathcal{T}_i \leq 3^{-i} N^3 (\log N)^{1/4}\} \cap \{\mathcal{T}_\infty \leq N^2 (\log N)^{1/4}\}, \end{aligned}$$

where

$$(6.44) \quad T := N^2 \sqrt{\log N}.$$

We assume also that  $N$  is large enough so that from Proposition 6.3 and Lemmas 6.6 and 6.8 we have

$$(6.45) \quad \mathbb{P}[\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}] \geq 1 - (2 \log \log N)^{-1}.$$

Hence, the results follows if we can prove that

$$(6.46) \quad \{\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}\} \subset \{\tau \leq T\}.$$

We split the proof of (6.46) in two statements. We want to show first that on the event  $\mathcal{A} \cap \mathcal{B} \cap \mathcal{C}$

$$(6.47) \quad \tau - \tau_{\lceil \log_2 N/2 \rceil} \leq (\log N)^{1/4} N^2,$$

and then that

$$(6.48) \quad \forall i \in \{0, \dots, \lceil (\log_2 N)/2 \rceil\}, \quad (\tau_i - \tau_{i-1}) \leq (i+1)^{-2} N^2 (\log N)^{1/3}.$$

These inequalities combined give

$$(6.49) \quad \tau \leq (\log N)^{1/4} N^2 + \sum_{i=0}^K (i+1)^{-2} N^2 (\log N)^{1/3} \leq N^2 \sqrt{\log N}.$$

Note that (6.47) is an immediate consequence of  $\mathcal{C}$  as

$$(6.50) \quad \mathcal{T}_\infty = \int_{\tau_K}^{\tau} u(t) dt \geq \tau - \tau_K.$$

Let us turn to (6.48). Let us assume that the statement is false and set

$$(6.51) \quad i_0 := \min\{i \in \{0, \dots, \lceil (\log_2 N)/2 \rceil\} \mid (\tau_i - \tau_{i-1}) > (i+1)^{-2} N^2 (\log N)^{1/3}\}.$$

The definition of  $i_0$  implies that

$$(6.52) \quad \tau_{i_0-1} + (i_0 + 1)^{-2} N^2 (\log N)^{1/3} \leq T.$$

From  $\mathcal{B}$ , we have [using (6.52) to obtain the second inequality]

$$(6.53) \quad \begin{aligned} & \int_{\tau_{i_0-1}}^{\tau_{i_0}} \mathbf{1}_{\{H(t) \leq \sqrt{N} \log \log N\}} \\ & \geq \int_{\tau_{i_0-1}}^{\tau_{i_0-1} + (i_0+1)^{-2} N^2 (\log N)^{1/3}} \mathbf{1}_{\{\mathcal{H}(t) \leq \sqrt{N} \log \log N\}} dt \\ & = (i_0 + 1)^{-2} N^2 (\log N)^{1/3} \\ & \quad - \int_{\tau_{i_0-1}}^{\tau_{i_0-1} + (i_0+1)^{-2} N^2 (\log N)^{1/3}} \mathbf{1}_{\{\mathcal{H}(t) > \sqrt{N} \log \log N\}} dt \\ & \geq (i_0 + 1)^{-2} N^2 (\log N)^{1/3} - N^2 (\log N)^{-3} \\ & \geq \frac{1}{2} (i_0 + 1)^{-2} N^2 (\log N)^{1/3}. \end{aligned}$$

For all  $t \leq \tau_{i_0}$ , we have  $A(t) \geq N^{3/2} 2^{-i_0}$ , and thus using Lemma 6.7 and the assumption that  $\mathcal{A}$  holds,

$$(6.54) \quad \begin{aligned} u(t) & \geq \frac{1}{8} \min\left(N, \frac{A(t)}{\max(\mathcal{H}(t), N^{1/2})}\right) \\ & \geq \frac{N^{3/2} 2^{-i_0}}{8\sqrt{N} \log \log N} \mathbf{1}_{\{\mathcal{H}(t) \leq \sqrt{N} \log \log N\}}. \end{aligned}$$

From (6.53),

$$(6.55) \quad \begin{aligned} \mathcal{T}_{i_0} & = \int_{\tau_{i_0-1}}^{\tau_{i_0}} u(t) dt \geq \frac{N 2^{-i_0}}{8(\log \log N)} \int_{\tau_{i_0-1}}^{\tau_{i_0}} \mathbf{1}_{\{\mathcal{H}(t) \leq \sqrt{N} \log \log N\}} dt \\ & \geq (i_0 + 1)^{-2} 2^{-i_0} \frac{N^3 (\log N)^{1/3}}{16 \log \log N} > 3^{-i_0} N^3 (\log N)^{1/4}. \end{aligned}$$

This brings a contradiction to  $\mathcal{C}$  (if  $N$  is large enough) and ends the proof of (6.46).  $\square$

**7. Proof of Proposition 4.3.** To prove the result, we will try to control the derivative in  $t$  of the total variation distance that we have to bound.

Note that  $\|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}}$  is always differentiable on the right. This comes from the fact that for each  $\eta \in \Omega_N$ , both  $\nu_t^\alpha(\eta)$  and  $\nu^{\alpha e^{-\lambda_N t}}$  are differentiable. With a small abuse of notation, we use  $\partial_t$  to denote the right derivative. Our method to prove Proposition 4.3 relies on getting a bound on the derivative valid for all  $\alpha \leq 1$ . More precisely, we want to prove

$$(7.1) \quad \partial_t \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} \leq C_1 \alpha^3 N^{-2} e^{-3\lambda_N t} + C_2 \alpha^2 N^{-3/2} e^{-2\lambda_N t}.$$

Indeed, once integrated this gives

$$(7.2) \quad \sup_{t \geq 0} \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} \leq C_3(\alpha^3 + \alpha^2 N^{1/2}),$$

which is equivalent to our result.

Let us first perform a simple computation to show that it is sufficient to prove (7.1) in the case  $t = 0$ . Using the triangular inequality and the fact that the semi-group shrinks the total-variation distance, we have for any positive  $\varepsilon$ ,

$$(7.3) \quad \begin{aligned} & \|\nu_{t+\varepsilon}^\alpha - \nu^{\alpha e^{-\lambda_N(t+\varepsilon)}}\|_{\text{TV}} \\ & \leq \|\nu_{t+\varepsilon}^\alpha - \nu_\varepsilon^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} + \|\nu_\varepsilon^{\alpha e^{-\lambda_N t}} - \nu^{\alpha e^{-\lambda_N(t+\varepsilon)}}\|_{\text{TV}} \\ & \leq \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} + \|\nu_\varepsilon^{\alpha e^{-\lambda_N t}} - \nu^{\alpha e^{-\lambda_N(t+\varepsilon)}}\|_{\text{TV}}. \end{aligned}$$

Hence,

$$(7.4) \quad \partial_t \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} \leq \partial_\varepsilon \|\nu_\varepsilon^{\alpha e^{-\lambda_N t}} - \nu^{\alpha e^{-\lambda_N(t+\varepsilon)}}\|_{\text{TV}}|_{\varepsilon=0}.$$

Note that the right-hand side is simply equal to

$$\partial_s \|\nu_s^{\alpha'} - \nu^{\alpha' e^{-\lambda_N s}}\|_{\text{TV}}|_{s=0}$$

for  $\alpha' = \alpha e^{-\lambda_N t}$ . Hence, to prove (7.1) it is sufficient to show that for all  $\alpha \leq 1$

$$(7.5) \quad \partial_t \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}}|_{t=0} \leq C_1 \alpha^3 N^{-2} + C_2 \alpha^2 N^{-3/2}.$$

We let  $g_t^\alpha$  denote the density of  $\nu_t^\alpha$ , and  $g^\alpha$  that of  $\nu^\alpha$ . Recall that we have

$$(7.6) \quad g^\alpha(\eta) := \frac{e^{\alpha a(\eta)}}{\mu_N(e^{\alpha a(\eta)})},$$

where

$$(7.7) \quad a(\eta) := \sum_{x \in \mathbb{Z}_{2N}} \eta(x) \overline{\sin}(x).$$

We have

$$(7.8) \quad \partial_t \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} = \mu_N |\partial_t (g_t^\alpha(\eta) - g^{\alpha e^{-\lambda_N t}}(\eta))|_{t=0}|.$$



We compute the derivatives of  $g_t^\alpha$  and  $g^{\alpha e^{-\lambda_N t}}(\eta)$  separately. We have

$$(7.9) \quad \partial_t g^{\alpha e^{-\lambda_N t}}(\eta)|_{t=0} = \alpha \lambda_N g^\alpha(\eta)[-a(\eta) + \nu^\alpha(a(\eta))].$$

The other term requires more work, and we have

$$(7.10) \quad \partial_t g_t^\alpha(\eta)|_{t=0} = \mathcal{L}g = \sum_{x \in \mathbb{Z}_{2N}} g^\alpha(\eta^x) - g^\alpha(\eta).$$

Recall (6.9). We have

$$(7.11) \quad g^\alpha(\eta^x) - g^\alpha(\eta) = g^\alpha(\eta)[\exp(-\alpha \nabla \overline{\sin}(x) \nabla \eta(x)) - 1].$$

Performing a Taylor expansion of the exponential, we have

$$(7.12) \quad \mathcal{L}g := g^\alpha(\eta) \left[ -\alpha \left( \sum_{x \in \mathbb{Z}_{2N}} \nabla \overline{\sin}(x) \nabla \eta(x) \right) + \frac{\alpha^2}{2} G(\eta, N) + R(\eta, N) \right],$$

where  $(\alpha^2/2)G(\eta, N)$  is the second term in the Taylor expansion

$$(7.13) \quad G(\eta, N) := \sum_{x \in \mathbb{Z}_{2N}} (\nabla \overline{\sin}(x))^2 (\nabla \eta(x))^2,$$

and  $R(\eta, N)$  is the Taylor rest

$$(7.14) \quad R(\eta, N) := \sum_{x \in \mathbb{Z}_{2N}} \left( e^{-\alpha \nabla \overline{\sin}(x) \nabla \eta(x)} - 1 + \alpha \nabla \overline{\sin}(x) \nabla \eta(x) - \frac{\alpha^2}{2} (\nabla \overline{\sin}(x))^2 (\nabla \eta(x))^2 \right).$$

The first term in the RHS of (7.12) can be simplified using summation by part and the fact that  $\overline{\sin}$  is an eigenfunction of  $\Delta$ . We have

$$(7.15) \quad \sum_{x \in \mathbb{Z}_{2N}} \nabla \overline{\sin}(x) \nabla \eta(x) = - \sum_{x \in \mathbb{Z}_{2N}} \Delta \overline{\sin}(x) \eta(x) = \lambda_N a(\eta).$$

According to Taylor's formula, one has for all  $\alpha < 1$ , for an adequate choice of constant  $C_1$

$$(7.16) \quad |R(\eta, N)| \leq \frac{e^{2\alpha} \alpha^3}{6} \sum_{x \in \mathbb{Z}_{2N}} |\nabla \overline{\sin}(x) \nabla \eta(x)|^3 \leq C_1 \alpha^3 N^{-2},$$

where in the last inequality we have used that  $|\nabla \eta(x)| \leq 2$  and that

$$(7.17) \quad |\nabla \overline{\sin}(x)| = 2 \left| \sin\left(\frac{\pi}{2N}\right) \cos\left(\frac{\pi x}{N} + \frac{\pi}{2N}\right) \right| \leq \frac{\pi}{N}.$$

Combining (7.8) with (7.9) and (7.15), we obtain

$$\begin{aligned}
 \partial_t \|\nu_t^\alpha - \nu^{\alpha e^{-\lambda_N t}}\|_{\text{TV}} &\leq \nu_\alpha \left| \frac{\alpha^2}{2} G(\eta, N) + R(\eta, N) - \alpha \lambda_N \nu^\alpha(a(\eta)) \right| \\
 (7.18) \quad &\leq \nu_\alpha \left| R(\eta, N) - \alpha \lambda_N \nu^\alpha(a(\eta)) + \frac{\alpha^2}{2} \nu_\alpha(G(\eta, N)) \right| \\
 &\quad + \frac{\alpha^2}{2} \nu^\alpha |G(\eta, N) - \nu^\alpha(G(\eta, N))|.
 \end{aligned}$$

To estimate the first term, we note that as

$$(7.19) \quad \mu_N(\partial_t(g_t^\alpha(\eta) - g^{\alpha e^{-\lambda_N t}}(\eta))|_{t=0}) = 0,$$

we have

$$(7.20) \quad \nu_\alpha \left( R(\eta, N) - \alpha \lambda_N \nu^\alpha(a(\eta)) + \frac{\alpha^2}{2} G(\eta, N) \right) = 0.$$

Hence, from (7.16)

$$\begin{aligned}
 (7.21) \quad &\nu_\alpha \left| R(\eta, N) - \alpha \lambda_N \nu^\alpha(a(\eta)) + \frac{\alpha^2}{2} \nu_\alpha(G(\eta, N)) \right| \\
 &= \nu_\alpha |R(\eta, N) - \nu^\alpha(R(\eta, N))| \leq C_1 \alpha^3 N^{-2}.
 \end{aligned}$$

To estimate the second term, we use Proposition B.1. The reader can check that the Lipschitz norm of  $G$  [cf. (B.1)] of  $G$  satisfies

$$(7.22) \quad \|G(\cdot, N)\|_{\text{lip}} \leq 8\pi^2 N^{-2}$$

and hence that for an adequate choice of  $C_2 > 0$

$$(7.23) \quad \nu^\alpha |G(\eta, N) - \nu^\alpha(G(\eta, N))| \leq C_2 N^{-3/2}.$$

This completes the proof of (7.5).

## APPENDIX A: PROOF OF TECHNICAL STATEMENTS ON $\nu^\alpha$ .

**A.1. Proof of Proposition 6.1.** Note that if  $\mu_N$  was replaced by the uniform measure on  $\{-1, 1\}^{\mathbb{Z}_{2N}}$  (without the constraint of having  $N$  particles) then  $\nu^\alpha$  would be a product of independent Bernoulli, and the statement would be trivial to prove.

What we have to control is that the constraint on the number of particles does not affect the mean too much. To do so, we perform an expansion of the partition function according to the value of  $\eta(x)$  to show that the ratio of the partition function restricted to the event  $\eta(x) = +1$  and  $\eta(x) = -1$ , respectively, is close to  $\exp(2\alpha \overline{\sin}(x))$ . To this purpose, we introduce the

quantity

$$(A.1) \quad Z(x) := \frac{\mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N} \setminus \{x\}} \eta(y) \overline{\sin}(y)} | \eta(x) = +1)}{\mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N} \setminus \{x\}} \eta(y) \overline{\sin}(y)} | \eta(x) = -1)}.$$

We have

$$(A.2) \quad \begin{aligned} & \nu^\alpha(\eta(x)) \\ &= \nu^\alpha(\eta(x) = +1) - \nu^\alpha(\eta(x) = -1) \\ &= \frac{\mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N}} \eta(y) \overline{\sin}(y)} | \eta(x) = +1) - \mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N}} \eta(y) \overline{\sin}(y)} | \eta(x) = -1)}{\mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N}} \eta(y) \overline{\sin}(y)} | \eta(x) = +1) + \mu_N(e^{\alpha \sum_{y \in \mathbb{Z}_{2N}} \eta(y) \overline{\sin}(y)} | \eta(x) = -1)} \\ &= \frac{e^{2\alpha \overline{\sin}(x)} Z(x) - 1}{e^{2\alpha \sin(x)} Z(x) + 1}. \end{aligned}$$

Hence, what we must check to prove (6.1) is that  $Z(x)$  is very close to one. Now note that we can obtain a coupling of  $\mu_N(\cdot | \eta(x) = -1)$  and  $\mu_N(\cdot | \eta(x) = +1)$  in the following manner: take  $\eta^1$  with distribution  $\mu_N(\cdot | \eta(x) = -1)$ , choose  $y$  uniformly at random (and independent of  $\eta^1$  in  $\{z | \eta^1(z) = +1\}$  and let  $\eta^2$  be obtained from  $\eta^1$  by exchanging the value at  $x$  and  $y$  (which are  $+1$  and  $-1$ , resp.). A consequence of this coupling is that

$$(A.3) \quad \begin{aligned} & Z(x) \\ &:= \frac{(1/N) \mu_N(\sum_{y \in \{z | \eta(z) = +1\}} e^{\alpha \sum_{w \in \mathbb{Z}_{2N} \setminus \{x\}} \eta(w) \overline{\sin}(w) - 2\alpha \overline{\sin}(y)} | \eta(x) = -1)}{\mu_N(e^{\alpha \sum_{w \in \mathbb{Z}_{2N} \setminus \{x\}} \eta(w) \overline{\sin}(w)} | \eta(x) = -1)}, \end{aligned}$$

and hence we can deduce from it

$$(A.4) \quad \begin{aligned} Z(x) &= \nu^\alpha \left( \frac{1}{N} \sum_{y \in \mathbb{Z}_{2N}} \frac{1 + \eta(y)}{2} e^{-2\alpha \overline{\sin}(y)} \middle| \eta(x) = -1 \right) \\ &= 1 + \nu^\alpha \left( \frac{1}{N} \sum_{y \in \mathbb{Z}_{2N}} \frac{1 + \eta(y)}{2} (e^{-2\alpha \overline{\sin}(y)} - 1) \middle| \eta(x) = -1 \right). \end{aligned}$$

Note that with this expression it is not hard to check that  $|Z(x) - 1| \leq e^{2\alpha} - 1$ . However, to get a sharper estimate, we must have a good control on  $\nu^\alpha(\eta(y) | \eta(x) = -1)$ . We obtain it by pushing the expansion one step further. We set

$$(A.5) \quad Z'(x, y) = \frac{\mu_N(e^{\alpha \sum_{z \in \mathbb{Z}_{2N} \setminus \{x, y\}} \overline{\sin}(z)} | \eta(x) = +1, \eta(y) = -1)}{\mu_N(e^{\alpha \sum_{z \in \mathbb{Z}_{2N} \setminus \{x, y\}} \overline{\sin}(z)} | \eta(x) = -1, \eta(y) = -1)}.$$

Similar to (A.2), we obtain that

$$(A.6) \quad \nu^\alpha(\eta(y)|\eta(x) = -1) = \frac{N}{N-1} \frac{e^{2\alpha\overline{\sin}(y)} Z'(x, y) - 1}{e^{2\alpha\overline{\sin}(y)} Z'(x, y) + 1}.$$

Like for (A.4), we have an alternative expression for  $Z'$

$$(A.7) \quad \begin{aligned} & Z'(x, y) \\ &= 1 + \nu^\alpha \left( \frac{1}{N} \sum_{z \in \mathbb{Z}_{2N}} \frac{1 + \eta(z)}{2} (e^{-2\alpha\overline{\sin}(z)} - 1) \middle| \eta(x) = -1, \eta(y) = -1 \right). \end{aligned}$$

Hence, we have

$$(A.8) \quad |Z'(x, y) - 1| \leq e^{2\alpha} - 1,$$

and from (A.6), we deduce that for some positive constant  $C_1$

$$(A.9) \quad |\nu^\alpha(\eta(y)|\eta(x) = -1)| \leq C_1 \left( \frac{1}{N} + \alpha \right).$$

Hence, we have

$$(A.10) \quad \begin{aligned} |Z(x) - 1| &\leq \frac{1}{2N} \left| \sum_{y \in \mathbb{Z}_{2N} \setminus \{x\}} (e^{-2\alpha\overline{\sin}(y)} - 1) \right| \\ &+ \frac{1}{2N} \sum_{y \in \mathbb{Z}_{2N} \setminus \{x\}} |\nu^\alpha(\eta(y)|\eta(x) = -1)(e^{-2\alpha\overline{\sin}(y)} - 1)|. \end{aligned}$$

Performing a Taylor expansion up to the second order in  $\alpha$  we obtain (recall  $\alpha \leq 1$ )

$$(A.11) \quad \left| \sum_{y \in \mathbb{Z}_{2N} \setminus \{x\}} (e^{-2\alpha\overline{\sin}(y)} - 1) \right| \leq 2\alpha|\overline{\sin}(x)| + \frac{eN\alpha^2}{2}.$$

The second term in the RHS of (A.10) can be bounded by

$$(A.12) \quad C_1 \left( \frac{1}{N} + \alpha \right) (e^\alpha - 1).$$

Hence, we obtain

$$(A.13) \quad |Z(x) - 1| \leq C_2(\alpha^2 + N^{-2}).$$

And then the result can easily be deduced from (A.2).

**A.2. Proof of Proposition 6.2.** The result follows from the combination of Proposition 5.1 which controls the fluctuation around the mean value  $u^{\eta_0}(x, t)$  given an initial condition  $\eta_0$  and the following statement, that the mean itself  $u^{\eta_0}(x, t)$  does not fluctuate too much if  $\eta_0$  has distribution  $\nu^\alpha$ .

LEMMA A.1. *There exists a constant  $c$  such that for all  $N > 0$ , for all  $|\alpha| \leq N^{-1/4}$ , and  $t \geq 0$*

$$(A.14) \quad \nu^\alpha \left[ \exists x, y \in \mathbb{Z}_{2N}, \left| \sum_{z=x+1}^y (u^\eta(x, t) - \alpha e^{-\lambda_N t} \overline{\sin}(z)) \right| \geq s\sqrt{N} \right] \leq 2e^{-cs^2}.$$

PROOF. It is in fact sufficient to prove (A.14) for  $t = 0$ , because

$$(A.15) \quad \max_{x, y} \sum_{z=x+1}^y (u^\eta(z, t) - \alpha e^{-\lambda_N t} \overline{\sin}(z)) \leq \max_{x, y} \sum_{z=x+1}^y (\eta_0(z) - \alpha \overline{\sin}(z)).$$

Indeed, if one sets  $v(x, t)$  to be the solution of the discrete-heat equation on  $\mathbb{Z}_{2N}$  with initial condition

$$v_0(x) := \sum_{z=1}^x \eta(z) - \alpha \overline{\sin}(z),$$

then (A.15) can be reformulated as

$$(A.16) \quad \max_{x, y} [v(t, y) - v(t, x)] \leq \max_{x, y} [v_0(y) - v_0(x)]$$

which is obviously true by contractivity of the heat equation. Note that at the cost of losing a factor in the constant  $c$ , we can restrict ourselves to proving that

$$(A.17) \quad \nu^\alpha \left[ \exists y \in \mathbb{Z}_{2N}, \left| \sum_{z=1}^y [\eta(z) - \nu^\alpha(\eta(z))] \right| \geq 4s\sqrt{N} \right] \leq 2e^{-cs^2}.$$

We have used Proposition 6.1 and the assumption on  $\alpha$  to replace  $\alpha \overline{\sin}(z)$  by  $\nu^\alpha(\eta(z))$ . Let us introduce notation for the sum

$$(A.18) \quad S_{x, y} := \sum_{z=1}^y (\eta(z) - \nu^\alpha(\eta(z))).$$

We also set  $p := \lfloor \log_2 N \rfloor + 1$ . For  $s > 0$ , we set

$$(A.19) \quad \mathcal{J}(s) := \{ \exists q \in \{1, \dots, p\}, \exists y \in \{1, \dots, \lfloor 2N2^{-q} \rfloor \}, \\ |S_{2^q(y-1), 2^q y}| \geq (\frac{3}{4})^{p-q} s\sqrt{N} \}.$$

By a simple dichotomy argument (see the proof of Proposition 4.1 in [9]), we have

$$(A.20) \quad \left\{ \exists y \in \mathbb{Z}_{2N}, \left| \sum_{z=1}^y (\eta(z) - 1/2 - \alpha \overline{\sin}(z)) \right| \geq 4s\sqrt{N} \right\} \subset \mathcal{J}(s).$$

For  $y$  and  $p$  fixed,  $S_{2^q(y-1), 2^q y}(\eta)$  is a function which depends on  $2^q$  coordinates and whose Lipschitz norm is smaller than 2. Hence, by Proposition B.1, we have

$$(A.21) \quad \begin{aligned} \nu^\alpha(|S_{2^q(y-1), 2^q y}| \geq (\tfrac{3}{4})^{p-q} s \sqrt{N}) &\leq 2 \exp(-C_1 (\tfrac{9}{16})^{p-q} 2^{-q} s^2 N) \\ &\leq 2 \exp(-2C_1 (\tfrac{9}{8})^{p-q} s^2). \end{aligned}$$

Hence, by a union bound, for an appropriate choice of constant  $C_2$  and for all  $s > 0$ , we have

$$(A.22) \quad \nu^\alpha(\mathcal{H}(s)) \leq 2 \sum_{q=1}^p 2^{p-q} \exp\left(-2C_1 \left(\frac{9}{8}\right)^{p-q} s^2\right) \leq 2 \exp(-C_2 s^2). \quad \square$$

### A.3. Proof of Proposition 6.3. Set

$$(A.23) \quad \bar{\mathcal{E}} := \{\eta \in \Omega_N \mid \exists (x, y) \in \mathbb{Z}_{2N}^2, \# [x, y] \geq N^{1/4} \Rightarrow j(x, y, \eta) \geq \tfrac{1}{3} \# [x, y]\}.$$

First, we notice that from the proof of [8], Lemma 6.2, there exists a constant  $C_1 > 0$  such that

$$(A.24) \quad \mu_N(\bar{\mathcal{E}}) \leq e^{-C_1 N^{1/4}}.$$

Recall that  $\nu_t^\alpha$  the law of  $\eta_t$  starting from distribution  $\nu^\alpha$ . We have by the Cauchy–Schwarz inequality

$$(A.25) \quad (\nu_t^\alpha(\bar{\mathcal{E}}))^2 \leq \mu_N(\bar{\mathcal{E}}) \mu_N \left[ \left( \frac{d\nu_t^\alpha}{d\mu_N} \right)^2 \right].$$

Note that the term  $\mu_N[(\frac{d\nu_t^\alpha}{d\mu_N})^2]$  is decreasing in  $t$ , because the semi-group of the Markov chain contracts the  $l_2$  norm. For  $t = 0$ , we have

$$(A.26) \quad \mu_N \left[ \left( \frac{d\nu^\alpha}{d\mu_N} \right)^2 \right] \leq \mu_N(e^{2\alpha a(\eta)}).$$

Using Proposition B.1 to have Gaussian concentration for  $a(\theta)$ , we have for  $N$  sufficiently large:

$$(A.27) \quad \mu_N \left[ \left( \frac{d\nu^\alpha}{d\mu_N} \right)^2 \right] \leq \exp(100N\alpha^2).$$

Hence, we can conclude that there exists constant  $C_2$  and  $C_3$  such that if  $\alpha < C_2 N^{-3/8}$  for any  $t$  we have

$$(A.28) \quad \nu_t^\alpha(\bar{\mathcal{E}}) = \mathbb{P}^{\nu^\alpha}[\eta_t \in \bar{\mathcal{E}}] \leq e^{-C_3 N^{1/4}}.$$

Now we have to move from this result to a result for all  $t \leq N^3$ . Note that starting from  $\eta \notin \bar{\mathcal{E}}$ , one needs at least  $\frac{1}{12}N^{1/4}$  transitions in order to jump out of  $\mathcal{E}$ . Hence, using union bound

$$(A.29) \quad \begin{aligned} & \mathbb{P}^{\nu^\alpha}[\exists t \leq N^3, \eta_t \notin \mathcal{E}] \\ & \leq \sum_{i=0}^{N^5} \mathbb{P}^{\nu^\alpha}[\eta_{i/N^2} \in \bar{\mathcal{E}}] + \sum_{i=1}^{N^5} \mathbb{P}\left[(\eta_t)_{t \in [(i-1)/N^2, i/N^2]} \right. \\ & \quad \left. \text{performs more than } \frac{1}{12}N^{1/4} \text{ transitions} \right]. \end{aligned}$$

The first term is smaller than  $e^{-C_3 N^{1/4}}$ ; cf. (A.28). As for the second one, it is not difficult to check that the rate at which transitions occur in the chain is bounded by  $2N$ , and thus that for any  $i$

$$(A.30) \quad \mathbb{P}[(\eta_t)_{t \in [(i-1)/N^2, i/N^2]} \text{ performs more than } \frac{1}{12}N^{1/4} \text{ transitions}] \leq e^{-N},$$

provided  $N$  is large enough.

## APPENDIX B: CONCENTRATION FOR LIPSCHITZ FUNCTION OF PARTICLE SYSTEMS.

Given  $f : \{0, 1\}_{2N}^{\mathbb{Z}} \rightarrow \mathbb{R}$ , one sets  $\|f\|_{\text{lip}}$  to be the Lipschitz norm of  $f$  for the Hamming distance

$$(B.1) \quad \|f\|_{\text{lip}} := \max_{\eta, \eta' \in \{-1, 1\}^{\mathbb{Z}_{2N}}} \frac{|f(\eta) - f(\eta')|}{\sum_{x \in \mathbb{Z}_{2N}} \mathbf{1}_{\{\eta(x) \neq \eta'(x)\}}}.$$

PROPOSITION B.1. *For any  $f : \{-1, 1\}^{\mathbb{Z}_{2N}} \rightarrow \mathbb{R}$  we have*

$$(B.2) \quad \mu_N(|f - \mu_N(f)| \geq s) \leq 2 \exp\left(-\frac{s^2}{8(2N-1)\|f\|_{\text{lip}}^2}\right).$$

*If the function  $f$  only depends on  $(\eta_x)_{x \in A}$  where  $A$  is fixed a subset of  $\mathbb{Z}_{2N}$  of cardinal  $k$  we have*

$$(B.3) \quad \mu_N(|f - \mu_N(f)| \geq s) \leq 2 \exp\left(-\frac{s^2}{8k\|f\|_{\text{lip}}^2}\right).$$

*The result remains valid if  $\mu_N$  is replaced by a measure  $\nu$  whose density with respect to  $\mu_N$  is of the form*

$$(B.4) \quad \frac{d\nu}{d\mu_N} := \frac{e^{\sum_{x \in \mathbb{Z}_{2N}} g(x)\eta(x)}}{\mu_N(e^{\sum_{x \in \mathbb{Z}_{2N}} g(x)\eta(x)})},$$

where  $g$  is an arbitrary function on  $\mathbb{Z}_{2N}$ .

PROOF. We can without loss of generality assume that  $\|f\|_{\text{lip}} = 1$ . Now, we introduce the martingale  $(M_i)_{i=0}^{2N-1}$  defined by

$$(B.5) \quad M_i(\eta) := \nu(f(\eta) | (\eta(x))_{x=1}^i).$$

We are going to check that the increments of  $M$  are bounded, that is,

$$(B.6) \quad \forall i \in \{0, \dots, 2N-2\}, \quad |M_{i+1} - M_i| \leq 2$$

and the proposition is then simply a consequence of Azuma's concentration inequality [1].

To check (B.6), we need to show that for any realization  $(\eta(x))_{x=1}^i$  one can couple  $\eta^1$  and  $\eta^2$  with law

$$(B.7) \quad \begin{aligned} \nu_1 &:= \nu(\cdot | (\eta(x))_{x=1}^i, \eta(i+1) = 1), \\ \nu_2 &:= \nu(\cdot | (\eta(x))_{x=1}^i, \eta(i+1) = -1) \end{aligned}$$

in a way that  $(\eta^1 - \eta^2)(x)$  has only two discrepancies, one at  $i+1$  and another one in where  $\eta^1(x) = 1 - \eta^2(x) = 0$ .

Note that  $\nu_1$  and  $\nu_2$  can be considered as a measure on  $\{-1, 1\} \rightarrow \{i+2, \dots, 2N\}$ , one which is concentrated on the set of configurations with  $k := N - \sum_{x=1}^i \eta(x) - 1$  particles, and the other on the set of configuration with  $k+1$  particles. What one can do is to first draw  $\eta^1$  according to  $\nu_1$ , and then add a 1 chosen at random to the configuration to obtain  $\eta^2$ . One  $\eta^1$  is given, and we choose at random a site  $X$  in  $\{x \in \{i+2, \dots, 2N\} | \eta^1(x) = -1\}$  with distribution

$$(B.8) \quad \frac{e^{g(x)}}{\sum_{\{x \in \{i+2, \dots, 2N\} | \eta^1(x) = -1\}} e^{g(x)}}.$$

One can check that  $\eta^2$  defined by

$$(B.9) \quad \eta^2(x) := \eta^1(x) + \mathbf{1}_{\{X=x\}} - \mathbf{1}_{\{x=i+1\}},$$

has distribution  $\nu_2$ .

For the case where  $f$  depends only on  $\eta|_A$ , we can consider a  $k$ -step martingale which unveils at each step the state  $\eta(x)$  of one  $x \in A$ .  $\square$

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